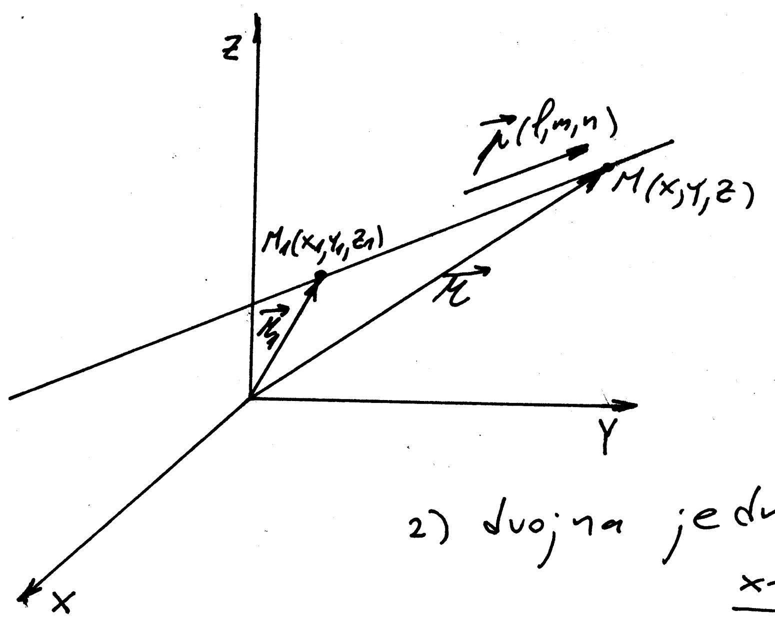
Prava u prostoru



Prava koja prolazi kroz tačku $M_1(x_1, y_1, z_1)$ i koja ima vektor pravca $\vec{p} = (l, m, n)$ ima sljedeće jednačine:

1) vektorska jednačina

$$(\vec{r} - \vec{r_1}) \times \vec{p} = 0$$

2) dvojna jednačina u kanoničkom obliku

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$$

3) parametarske jednačine

$$x = x_1 + lt$$
$$y = y_1 + mt$$
$$z = z_1 + nt$$

$$\begin{cases} A_1 x + B_1 y + C_1 z + D_1 = 0 \\ A_2 x + B_2 y + C_2 z + D_2 = 0 \end{cases}$$ jednačina prave koja je data presjekom dvije ravni

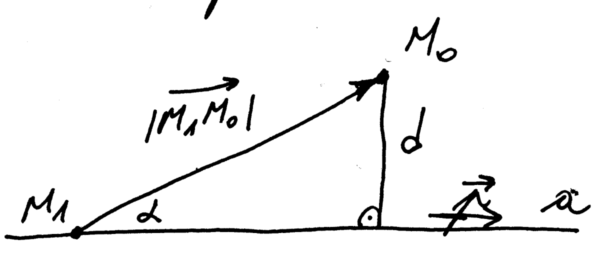
$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$ jednačina ravni kroz dvije tačke

Potreban uslov da se prave $a: \frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1}$ i $b: \frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2}$ sijeku jest da je

$$\begin{vmatrix} x_1-x_2 & y_1-y_2 & z_1-z_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

$$d = \frac{|(\vec{p_1} \times \vec{p_2}) \cdot \overrightarrow{M_1M_2}|}{|\vec{p_1} \times \vec{p_2}|}$$ udaljenost između dvije prave

\# Izvesti formulu za rastojanje tačke $M_0 \notin a$ od prave a.



$M_1 \in a$

Prava a ima vektor pravca $\vec{p}$

$$\sin\alpha = \frac{d}{|\overrightarrow{M_1M_0}|} \Rightarrow d = |\overrightarrow{M_1M_0}| \cdot \sin\alpha$$

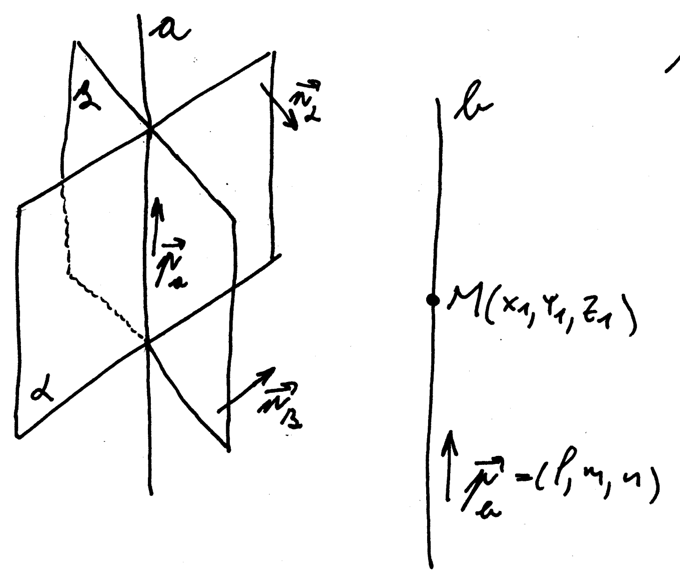
Od ranije znamo da je $|\vec{a}\times\vec{b}| = |\vec{a}|\,|\vec{b}|\cdot\sin\sphericalangle(\vec{a},\vec{b})$

pa ćemo imati $\sin(\vec{p}, \overrightarrow{M_1M_0}) = \dfrac{|\vec{p}\times\overrightarrow{M_1M_0}|}{|\vec{p}|\cdot|\overrightarrow{M_1M_0}|}$

dobićemo $d = \dfrac{|\vec{p}\times\overrightarrow{M_1M_0}|}{|\vec{p}|}$ rastojanje tačke M_0 od prave a

(#) Naći jednačinu prave koja sadrži tačku $M(-4,3,0)$ i paralelna je pravoj $\begin{cases} x-2y+z-4=0 \\ 2x+y-z=0 \end{cases}$.

Rj.



$b: \dfrac{x-x_1}{l} = \dfrac{y-y_1}{m} = \dfrac{z-z_1}{n}$

$\alpha: x-2y+z-4=0$

$\beta: 2x+y-z=0$

$\vec{n}_\alpha=(1,-2,1)$

$\vec{n}_\beta=(2,1,-1)$

$\left.\begin{array}{l} \vec{p}_a \perp \vec{n}_\alpha \\ \vec{p}_a \perp \vec{n}_\beta \end{array}\right\} \Rightarrow \vec{p}_a \parallel \vec{n}_\alpha\times\vec{n}_\beta$

$\Downarrow$

$\vec{p}_a = k(\vec{n}_\alpha\times\vec{n}_\beta)$, $k\in\mathbb{R}$

$\vec{p}_a \parallel \vec{p}_b \Rightarrow \vec{p}_b = t\cdot\vec{p}_a$, $t\in\mathbb{R}$

$$\vec{n}_\alpha\times\vec{n}_\beta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 1 \\ 2 & 1 & -1 \end{vmatrix} = \vec{i}\,(2-1)-\vec{j}\,(-1-2)+\vec{k}\,(1+4) =$$
$$= \vec{i}+3\vec{j}+5\vec{k} = (1,3,5)$$

$\vec{p}_b=(1,3,5)$

$M(-4,3,0)$

$\dfrac{x+4}{1} = \dfrac{y-3}{3} = \dfrac{z}{5}$ jednačina prave koja sadrži tačku M i paralelna je pravoj

(#) Odrediti λ u jednačini prave $\frac{x-3}{1}=\frac{y-1}{\lambda}=\frac{z+2}{1}$ da bi se sjekla sa pravom $\frac{x-1}{2}=\frac{y-2}{1}=\frac{z-1}{-1}$ i u tom slučaju naći presječnu tačku i ugao između pravih.

Rj:

$$a: \frac{x-3}{1}=\frac{y-1}{\lambda}=\frac{z+2}{1}, \quad \vec{p}_a=(1,\lambda,1), \quad x_1=3, \quad y_1=1, \quad z_1=-2$$

$$b: \frac{x-1}{2}=\frac{y-2}{1}=\frac{z-1}{-1}, \quad \vec{p}_b=(2,1,-1), \quad x_2=1, \quad y_2=2, \quad z_2=1$$

Potreban uslov da se prave sijeku: $\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$.

$$\begin{vmatrix} -2 & 1 & 3 \\ 1 & \lambda & 1 \\ 2 & 1 & -1 \end{vmatrix} \; \frac{I_R+III_R\cdot 3}{II_R+III_R} \; \begin{vmatrix} 4 & 4 & 0 \\ 3 & \lambda+1 & 0 \\ 2 & 1 & -1 \end{vmatrix} = (-1)\begin{vmatrix} 4 & 4 \\ 3 & \lambda+1 \end{vmatrix} = (-1)(4\lambda+4-12)=(-1)(4\lambda-8)$$

$$(-1)(4\lambda-8)=0$$
$$\lambda=2$$

Za vrijednost $\lambda=2$ prave a i b se sijeku.

a:

$$\frac{x-3}{1}=\frac{y-1}{2}=\frac{z+2}{1} \quad (=t)$$

$x-3=t \qquad x=t+3$
$y-1=2t \qquad y=2t+1$
$z+2=t \qquad z=t-2$

b:

$$\frac{x-1}{2}=\frac{y-2}{1}=\frac{z-1}{-1} \quad (=s)$$

$x-1=2s \qquad x=2s+1$
$y-2=s \qquad y=s+2$
$z-1=-s \qquad z=-s+1$

$t+3=2s+1$
$2t+1=s+2$
$t-2=-s+1$

$t-2s=-2 \quad /\cdot 2$
$2t-s=1$
$t+s=3 \quad /\cdot 2$

$2t-4s=-4 \quad (1)$
$2t-s=1 \quad (2)$
$2t+2s=6 \quad (3)$

$(1)-(3): \; -6s=-10$
$(2)-(3): \; -3s=-5$

$s=\frac{5}{3}$

$$t=2s-2=\frac{10}{3}-\frac{6}{3}=\frac{4}{3} \qquad x=\frac{4}{3}+3=\frac{13}{3}, \quad y=\frac{8}{3}+1=\frac{11}{3}, \quad z=\frac{4}{3}-2=-\frac{2}{3}$$

Presječna tačka pravih je $M\left(\frac{13}{3}, \frac{11}{3}, -\frac{2}{3}\right)$.

$$\vec{p}_a\cdot\vec{p}_b=(1,2,1)\cdot(2,1,-1)=2+2-1=3$$

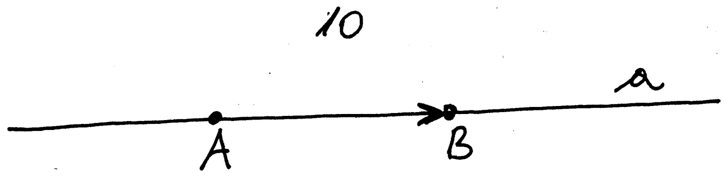
$$|\vec{p}_a|=\sqrt{1+4+1}=\sqrt{6}, \quad |\vec{p}_b|=\sqrt{4+1+1}=\sqrt{6} \qquad \vec{p}_a\cdot\vec{p}_b=|\vec{p}_a|\,|\vec{p}_b|\cdot\cos\sphericalangle(\vec{p}_a,\vec{p}_b)$$

$$\Rightarrow \cos\sphericalangle(\vec{p}_a,\vec{p}_b)=\frac{\vec{p}_a\cdot\vec{p}_b}{|\vec{p}_a|\cdot|\vec{p}_b|}=\frac{3}{6}=\frac{1}{2} \Rightarrow \sphericalangle(\vec{p}_a,\vec{p}_b)=60^\circ$$ ugao između pravih

Na pravoj $\frac{x-8}{8} = \frac{y-2}{-6} = \frac{z}{0}$ naći tačku čije rastojanje od tačke $A(8,2,0)$ iznosi 10.

Rj. $a: \frac{x-8}{8} = \frac{y-2}{-6} = \frac{z}{0} \quad (=t)$ $\qquad A(8,2,0)$

Tačka A pripada pravoj a.



$$a: \begin{cases} x = 8t + 8 \\ y = -6t + 2 \\ z = 0 \end{cases}$$

Tražimo tačku B tako da je $|\overrightarrow{AB}| = 10$

$B(8t+8, -6t+2, 0)$

$\overrightarrow{AB} = (8t, -6t, 0)$

$|\overrightarrow{AB}| = \sqrt{64t^2 + 36t^2}$

$|\overrightarrow{AB}| = 10$

$\sqrt{100t^2} = 10$

$10|t| = 10$

$|t| = 1$

$t_1 = -1 \quad t_2 = 1$

$B_1(0,8,0)$

$B_2(16,-4,0)$

Tačke B_1 i B_2 su tražene tačke

Naći rastojanje između ravni $\alpha: x - 2y + z - 1 = 0$ i ravni $\beta: 2x - 4y + 2z + 1 = 0$.

Napisati jednačinu ravni koja prolazi kroz tačke $P(1,1,1)$, $Q(0,1,-1)$ i normalna je na ravan α: $x + y + z - 1 = 0$.

Odrediti jednačinu ravni koja je paralelna sa vektorima $\overrightarrow{PQ}$ i $\overrightarrow{RT}$ i prolazi kroz tačku $M(0,1,0)$ ako su $P(-3,-2,-2)$, $Q(0,0,2)$, $R(-3,1,0)$ i $T(1,2,2)$.

Odrediti uglove koje obrazuju prava

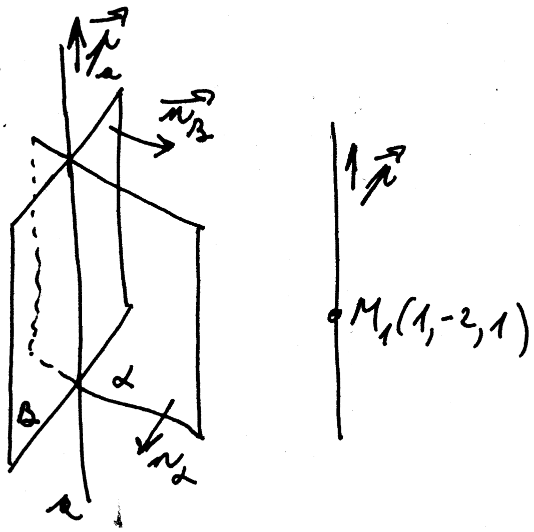
$a: \begin{cases} 2x - 2y - z - 8 = 0 \\ x + 2y - 2z - 4 = 0 \end{cases}$ i prava $b: \begin{cases} 4x + y + 3z - 4 = 0 \\ 2x + 2y - 3z - 11 = 0 \end{cases}$.

Rj. $\cos\varphi = \frac{4}{21}$

Odrediti presječnu tačku pravih

$\begin{cases} 5x - 2y + 5z + 3 = 0 \\ x + 3y - 4z - 10 = 0 \end{cases}$ i $\begin{cases} 3x + 10y - z - 47 = 0 \\ 6x - 2y + 7z + 3 = 0 \end{cases}$ Rj. $(2,4,-1)$

(#) Kroz tačku $M_1(1,-2,1)$ povući pravu paralelnu pravoj $\begin{cases} x-y+z-4=0 \\ 2x+y-2z+5=0 \end{cases}$.

R: 

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$$ jednačina prave kroz tačku $M(x_1,y_1,z_1)$

$\alpha: x-y+z-4=0$

$\vec{n}_\alpha=(1,-1,1)$ vektor normale na ravan α

$\beta: 2x+y-2z+5=0$

$\vec{n}_\beta=(2,1,-2)$ vektor normale na ravan β

$\vec{p}_a \parallel \vec{p}$

$$\left.\begin{array}{l}\vec{p}_a \perp \vec{n}_\alpha \\ \vec{p}_a \perp \vec{n}_\beta\end{array}\right\} \Rightarrow \left.\begin{array}{l}\vec{p}_a \parallel \vec{n}_\alpha \times \vec{n}_\beta \\ \vec{p} \parallel \vec{p}_a\end{array}\right\} \Rightarrow \vec{p} \parallel \vec{n}_\alpha \times \vec{n}_\beta$$

$$\vec{n}_\alpha \times \vec{n}_\beta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 1 \\ 2 & 1 & -2 \end{vmatrix} = (2-1)\vec{i}-(-2-2)\vec{j}+(1+2)\vec{k}=(1,4,3)$$

Za vektor pravca tražene prave mogu uzeti

$$\vec{p}=(1,4,3)$$

$$M_1(\overset{x_1}{1},\overset{y_1}{-2},\overset{z_1}{1})$$

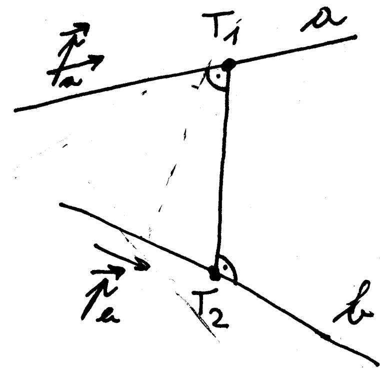
$$\frac{x-1}{1}=\frac{y+2}{4}=\frac{z-1}{3}$$

jednačina tražene prave

Izračunati rastojanje između pravih

$$\frac{x-1}{4}=\frac{y}{-3}=\frac{z+5}{-1} \quad i \quad \frac{x}{-3}=\frac{y+4}{2}=\frac{z-1}{6}.$$

Rj.



I način:

$a: \quad \frac{x-1}{4}=\frac{y}{-3}=\frac{z+5}{-1}=s$

$$\left.\begin{matrix} x-1=4s \\ y=-3s \\ z+5=-s \end{matrix}\right\} \Rightarrow \begin{matrix} x=4s+1 \\ y=-3s \\ z=-s-5 \end{matrix}$$

$b: \quad \frac{x}{-3}=\frac{y+4}{2}=\frac{z-1}{6}=t$

$$\left.\begin{matrix} x=-3t \\ y+4=2t \\ z-1=6t \end{matrix}\right\} \Rightarrow \begin{matrix} x=-3t \\ y=2t-4 \\ z=6t+1 \end{matrix}$$

$$\left.\begin{matrix} \overrightarrow{T_1T_2} \perp \vec{p}_a \\ \overrightarrow{T_1T_2} \perp \vec{p}_b \end{matrix}\right\} \Rightarrow \begin{matrix} \overrightarrow{T_1T_2}\cdot\vec{p}_a=0 \\ \overrightarrow{T_1T_2}\cdot\vec{p}_b=0 \end{matrix}$$

$$\begin{matrix} T_1(4s+1,\,-3s,\,-s-5) \\ T_2(-3t,\,2t-4,\,6t+1) \end{matrix} \Rightarrow \overrightarrow{T_1T_2}=(-3t-4s-1,\ 2t+3s-4,\ 6t+s+6)$$

$$d=|\overrightarrow{T_1T_2}|$$

$\vec{p}_a=(4,-3,-1)$

$\vec{p}_b=(-3,2,6)$

$$\vec{p}_a\times\vec{p}_b=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & -3 & -1 \\ -3 & 2 & 6 \end{vmatrix}=\vec{i}(-16)-\vec{j}(21)+\vec{k}(-1)$$

$$=-16\vec{i}-21\vec{j}-\vec{k}$$

$\vec{p}_a\times\vec{p}_b=(-16,-21,-1)$

$$\left.\begin{matrix} \vec{p}_a\cdot\overrightarrow{T_1T_2}=0 \\ \vec{p}_b\cdot\overrightarrow{T_1T_2}=0 \end{matrix}\right\} \Rightarrow \left.\begin{matrix} -12s-13t+1=0 \\ 49s+24t+31=0 \end{matrix}\right\} \Rightarrow \begin{matrix} s=\frac{-427}{349} \\ t=\frac{421}{349} \end{matrix}$$

$$\overrightarrow{T_1T_2}=\left(\frac{-752}{349},\ \frac{-987}{349},\ \frac{-47}{349}\right)=\left(\frac{-2^4\cdot 47}{349},\ \frac{-3\cdot 7\cdot 47}{349},\ \frac{-47}{349}\right)$$

$$d=|\overrightarrow{T_1T_2}|=\sqrt{\frac{4418}{349}}=\frac{94}{\sqrt{2\cdot 349}}=\frac{94}{\sqrt{698}}$$ rastojanje između pravih

II način

$|(\vec{p}_a\times\vec{p}_b)\cdot\overrightarrow{M_1M_2}|$ zapremina paralelopipeda $=V$

$|\vec{p}_a\times\vec{p}_a|$ površina paralelograma $=B$

$$V=B\cdot H$$

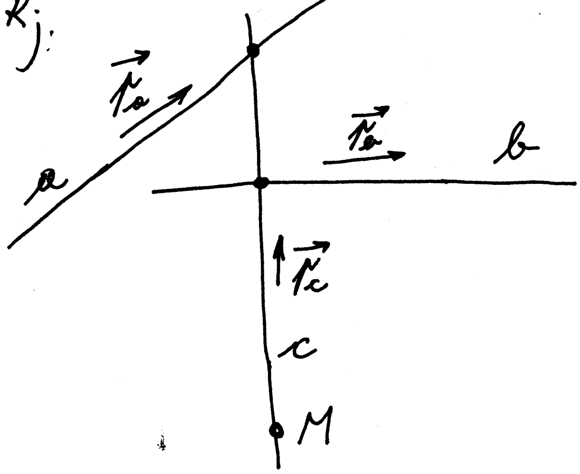
$$H=\frac{V}{B}$$

$$H=d=\frac{|(\vec{p}_a\times\vec{p}_a)\cdot\overrightarrow{M_1M_2}|}{|\vec{p}_a\times\vec{p}_a|}=\frac{94}{\sqrt{698}}$$

(#) Naći jednačinu prave koja prolazi kroz tačku M $M(0,2,-5)$ i siječe prave

$$a: \frac{x-1}{5} = \frac{y+1}{-1} = \frac{z+4}{7} \quad i \quad b: \frac{x+4}{2} = \frac{y-2}{4} = \frac{z+10}{2}.$$

Rj:



$c: ?$

$$c: \frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} \ (=t$$

$$\vec{p}_c = (l, m, n)$$

$$c: \begin{cases} x = lt + x_1 \\ y = mt + y_1 \\ z = nt + z_1 \end{cases}$$

$M(0,2,-5)$

$$\text{pa } c: \begin{cases} x = lt \\ y = mt + 2 \\ z = nt - 5 \end{cases}$$ parametarski oblik prave c

$$a: \frac{x-1}{5} = \frac{y+1}{-1} = \frac{z+4}{7} \ (=s)$$

$$b: \frac{x+4}{2} = \frac{y-2}{4} = \frac{z+10}{2} \ (=r)$$

$$\begin{aligned} x-1 &= 5s \\ y+1 &= -s \\ z+4 &= 7s \end{aligned} \Rightarrow a: \begin{cases} x = 5s+1 \\ y = -s-1 \\ z = 7s-4 \end{cases}$$

$$b: \begin{cases} x = 2r-4 \\ y = 4r+2 \\ z = 2r-10 \end{cases}$$ parametarski oblik prave b

Nađimo presječnu tačku pravih a i c.

$$\begin{aligned} 5s+1 &= lt \\ -s-1 &= mt+2 \\ 7s-4 &= nt-5 \end{aligned}$$

$$\begin{aligned} 5s - lt &= -1 \quad (1) \\ -s - mt &= 3 \quad (2) \\ 7s - nt &= -1 \quad (3) \end{aligned}$$

$(1)+5\cdot(2)$: $-lt - 5mt = 14$

$(3)+7\cdot(2)$: $-nt - 7mt = 20$

$(-l-5m)t = 14$

$(-n-7m)t = 20$

$$t = \frac{14}{-l-5m} = \frac{20}{-n-7m}$$

$-14n - 98m = -20l - 100m \quad /:2$

$10l - 7n + m = 0$

$10l + m - 7n = 0$

Pokušajmo naći presječnu tačku pravih b i c:

$$\begin{aligned} 2r-4 &= lt \quad /\cdot 2 \\ 4r+2 &= mt+2 \\ 2r-10 &= nt-5 \quad /\cdot 2 \end{aligned}$$

$$\begin{aligned} 4r &= 2lt+8 \\ 4r &= mt \\ 4r &= 2nt+10 \end{aligned}$$

$$\left.\begin{aligned} 2lt+8 &= mt \\ 2nt+10 &= mt \end{aligned}\right\}$$

$(2l-m)t = -8$

$(2n-m)t = -10$

$$t = \frac{-8}{2l-m}$$

$$t = \frac{-10}{2n-m}$$

$2lt+8 = 2nt+10$

$$t = \frac{1}{l-n}$$

Sad možemo formirati jednakost:

$$\frac{-8}{2l-m} = \frac{-10}{2n-m}$$

$-16n + 8m = -20l + 10m$

$20l - 2m - 16n = 0 \quad /:2$

$10l - m - 8n = 0$

$$\begin{aligned} 10l + m - 7n &= 0 \quad (I) \\ 10l - m - 8n &= 0 \quad (II) \end{aligned}$$

$(I)+(II)$: $20l - 15n = 0$

$l = \frac{3}{4}n$

$(I)-(II)$: $2m + n = 0$

$m = -\frac{1}{2}n$

$$\vec{p}_c = \left(\frac{3}{4}n, -\frac{1}{2}n, n\right)_{n\in R}$$

$$\frac{x}{3} = \frac{y-2}{-2} = \frac{z+5}{4}$$

jednačina tražene prave

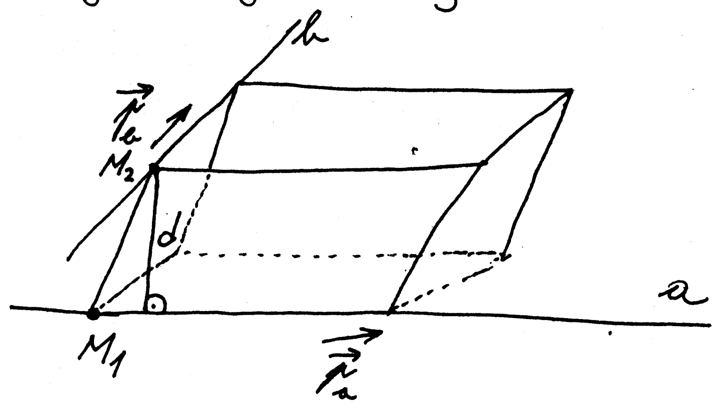
(#) Izračunati rastojanje između pravih

$$\frac{x-1}{9}=\frac{y}{2}=\frac{z+5}{-4} \quad i \quad \frac{x}{-6}=\frac{y+4}{-6}=\frac{z-1}{5}$$

Rj.

$$a: \frac{x-1}{9}=\frac{y}{2}=\frac{z+5}{-4}$$

$$b: \frac{x}{-6}=\frac{y+4}{-6}=\frac{z-1}{5}$$



$\vec{p}_a=(9,2,-4)$ $\quad M_1(1,0,-5)$

$\vec{p}_b=(-6,-6,5)$ $\quad M_2(0,-4,1)$

$\overrightarrow{M_1M_2}(-1,-4,6)$

Zapremina paralopipeda konstruisanog nad vektorima $\vec{p}_a, \vec{p}_b$ i $\overrightarrow{M_1M_2}$ računamo po formuli $|(\vec{p}_a \times \vec{p}_b)\cdot \overrightarrow{M_1M_2}|$.

Zapreminu paralopipeda možemo računati i po formuli $V=B\cdot H$ gdje je B površina paralelograma $|\vec{p}_a \times \vec{p}_b|$

$$H=\frac{V}{B} \quad tj. \quad d=\frac{|(\vec{p}_a\times\vec{p}_b)\cdot\overrightarrow{M_1M_2}|}{|\vec{p}_a\times\vec{p}_b|}$$ udaljenost između pravih

$$|(\vec{p}_a\times\vec{p}_b)\cdot\overrightarrow{M_1M_2}|=\begin{vmatrix}9&2&-4\\-6&-6&5\\-1&-4&6\end{vmatrix}\ \frac{II_k-I_k\cdot 4}{III_k+I_k\cdot 6}\ \begin{vmatrix}9&-34&50\\-6&18&-31\\-1&0&0\end{vmatrix}=(-1)\begin{vmatrix}-34&50\\18&-31\end{vmatrix}=$$

$$=(-1)\,2\begin{vmatrix}-17&25\\18&-31\end{vmatrix}=(-2)(527-450)=(-2)\cdot 77=-154$$

$$\vec{p}_a\times\vec{p}_b=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\9&2&-4\\-6&-6&5\end{vmatrix}=-14\vec{i}-21\vec{j}-42\vec{k}=(-14,-21,-42)$$

$$|\vec{p}_a\times\vec{p}_b|=\sqrt{14^2+21^2+42^2}=\sqrt{2^2\cdot 7^2+3^2\cdot 7^2+6^2\cdot 7^2}=7\sqrt{4+9+36}=7\cdot 7=49$$

udaljenost je uvijek pozitivna pa $d=\frac{154}{49}=\frac{22}{7}=3\frac{1}{7}$

tražena udaljenost

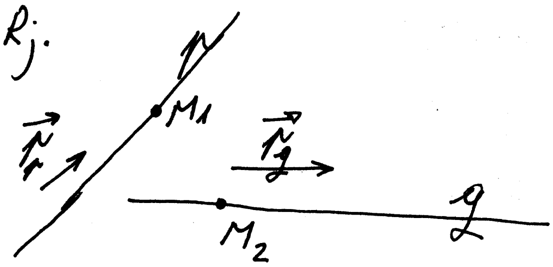
\# Date su prave $p: \dfrac{x-4}{1} = \dfrac{y+3}{2} = \dfrac{z-12}{-1}$;

$q: \dfrac{x-3}{-7} = \dfrac{y-1}{2} = \dfrac{z-1}{3}$.

a) Utvrditi međusobni položaj pravih p i q.

b) Naći jednačinu zajedničke normale pravih p i q.

Rj.



$\vec{p}_p = (1,2,-1)$

$\vec{p}_q = (-7,2,3)$

$M_1 \in p$

$M_1(4,-3,12)$

$M_2 \in q$

$M_2(3,1,1)$

$\overrightarrow{M_1M_2} = (-1,4,-11)$

Ako je $(\vec{p}_p \times \vec{p}_q) \cdot \overrightarrow{M_1M_2} = 0$

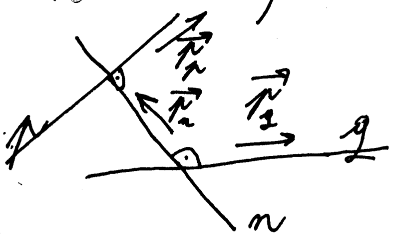
tada su prave p i q komplanarne (nalaze se u istoj ravni)

$$(\vec{p}_p \times \vec{p}_q) \cdot \overrightarrow{M_1M_2} = \begin{vmatrix} 1 & 2 & -1 \\ -7 & 2 & 3 \\ -1 & 4 & -11 \end{vmatrix} \overset{II_v - I_v}{\underset{III_v - I_v \cdot 2}{=\!=}} \begin{vmatrix} 1 & 2 & -1 \\ -8 & 0 & 4 \\ -3 & 0 & -9 \end{vmatrix} = (-2)\begin{vmatrix} -8 & 4 \\ -3 & -9 \end{vmatrix} =$$

$$= (-2)(-3)(4)\begin{vmatrix} -2 & 1 \\ 1 & 3 \end{vmatrix} = 6 \cdot 4 \cdot (-7) \neq 0$$

Prave p i q su dvije mimoilazne prave.

Nađimo zajedničku normalu n pravih p i q.



Za vektore pravca važi

$$\left.\begin{array}{l} \vec{p}_n \perp \vec{p}_p \\ \vec{p}_n \perp \vec{p}_q \end{array}\right\} \Rightarrow \vec{p}_n \parallel \vec{p}_p \times \vec{p}_q$$

$$\Downarrow$$

$\exists k \in \mathbb{R}$, $k \neq 0$ $\quad \vec{p}_n = k(\vec{p}_p \times \vec{p}_q)$

$$\vec{p}_p \times \vec{p}_q = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & -1 \\ -7 & 2 & 3 \end{vmatrix} = 8\vec{i} + 4\vec{j} + 16\vec{k} = 4(2,1,4)$$

$\vec{p}_n = 4k(2,1,4)$, k je neki broj

$n: \dfrac{x-x_1}{p} = \dfrac{y-y_1}{m} = \dfrac{z-z_1}{n}$ jednačina prave

$$\frac{x-x_1}{4k \cdot 2} = \frac{y-y_1}{4k} = \frac{z-z_1}{4k \cdot 4} \quad \Big/ \cdot 4k$$

Trebamo još naći tačku koja pripada pravoj n.

$$\frac{x-x_1}{2} = \frac{y-y_1}{1} = \frac{z-z_1}{4}$$

Da bi našli tačku $M(x_1, y_1, z_1)$ koja pripada pravoj n prvo ćemo pokušati naći presječne tačke pravih p i n i pravih q i n i na osnovu toga nešto zaključiti

$$p: \begin{cases} x = t+4 \\ y = 2t-3 \\ z = -t+12 \end{cases} \qquad q: \begin{cases} x = -7s+3 \\ y = 2s+1 \\ z = 3s+1 \end{cases} \qquad n: \begin{cases} x = 2r+x_1 \\ y = r+y_1 \\ z = 4r+z_1 \end{cases}$$

$p \cap n$:
$$t+4 = 2r+x_1 \quad (I)$$
$$2t-3 = r+y_1 \quad (II)$$
$$-t+12 = 4r+z_1 \quad (III)$$

$(I)+(III)$: $16 = 6r+x_1+z_1$

$(II)+2\cdot(III)$: $21 = 9r+y_1+2z_1$

$$r = \frac{16-x_1-z_1}{6} = \frac{21-y_1-2z_1}{9}$$

$$144-9x_1-9z_1 = 126-6y_1-12z_1$$
$$-9x_1+6y_1+3z_1+18 = 0 \quad /:3$$
$$-3x_1+2y_1+z_1+6 = 0$$

$q \cap n$:
$$-7s+3 = 2r+x_1 \quad (a)$$
$$2s+1 = r+y_1 \quad (b)$$
$$3s+1 = 4r+z_1 \quad (c)$$

$(a)-2(b)$: $-11s+1 = x_1-2y_1$

$(c)-4(b)$: $-5s-3 = z_1-4y_1$

$$s = \frac{1-x_1+2y_1}{11} = \frac{-3-z_1+4y_1}{5}$$

$$5-5x_1+10y_1 = -33-11z_1+44y_1$$
$$-5x_1-34y_1+11z_1+38 = 0$$

$$-3x_1+2y_1+z_1+6=0$$
$$-5x_1-34y_1+11z_1+38=0$$

$$z_1 = 3x_1-2y_1-6$$
$$-5x_1-34y_1+11z_1+38=0$$

$$-5x_1-34y_1+33x_1-22y_1-66+38=0$$
$$28x_1-56y_1-28=0 \quad /:28$$
$$x_1 = 2y_1+1$$

$$z_1 = 3x_1-2y_1-6$$
$$z_1 = 6y_1+3-2y_1-6$$
$$z_1 = 4y_1-3$$

Dobili smo da tačka M ima koordinate $M(2y_1+1, y_1, 4y_1-3)$.

Pokušajmo sad naći presječnu tačku pravih p i n

$$r = \frac{16-x_1-z_1}{6} = \frac{16-2y_1-1-4y_1+3}{6} = \frac{-6y_1+18}{6} = -y_1+3$$

$$t+4 = 2r+x_1 \Rightarrow t = 2(-y_1+3)+2y_1+1-4 = -2y_1+6+2y_1-3 = 3$$

$t=3$ Presječna tačka pravih p i n je $(7, 3, 9)$

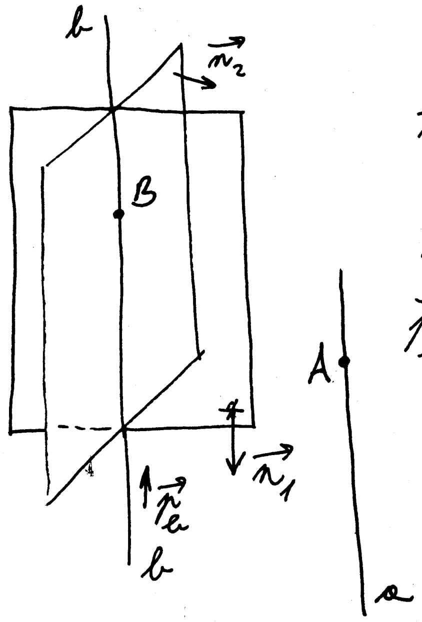
Za tačku M mogu uzeti koordinate $(7, 3, 9)$ pa

$$\frac{x-7}{2} = \frac{y-3}{1} = \frac{z-9}{4}$$ zajednička normala pravih p i q

(#) Nađi konstante α, β i γ tako da prava

$$a: \begin{cases} x = t+2 \\ y = -t-3 \\ z = \gamma t - 1 \end{cases} \quad \text{bude paralelna pravoj} \quad b: \begin{cases} \alpha x - 3y - z + 1 = 0 \\ x + \beta y + 2z - 4 = 0 \end{cases}$$

Rj.



$$\vec{n}_1 = (\alpha, -3, -1)$$
$$\vec{n}_2 = (1, \beta, 2)$$

$$\left.\begin{array}{l} \vec{p}_b \perp \vec{n}_1 \\ \vec{p}_b \perp \vec{n}_2 \end{array}\right\} \Rightarrow \vec{p}_b \parallel \vec{n}_1 \times \vec{n}_2$$
$$\Downarrow$$
$$\exists k \in \mathbb{R} \quad \vec{p}_b = k(\vec{n}_1 \times \vec{n}_2)$$

$$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \alpha & -3 & -1 \\ 1 & \beta & 2 \end{vmatrix} =$$
$$= (-6+\beta)\vec{i} - (2\alpha+1)\vec{j} + (\alpha\beta+3)\vec{k}$$
$$= (-6+\beta, -2\alpha-1, \alpha\beta+3)$$
$$\Downarrow$$
$$\vec{p}_b = k(-6+\beta, -2\alpha-1, \alpha\beta+3)$$
$k \in \mathbb{R}$, $k \neq 0$

$$a: \begin{cases} x = t+2 \\ y = -t-3 \\ z = \gamma t - 1 \end{cases} \Rightarrow \begin{array}{l} t = x-2 \\ -t = y+3 \\ \gamma t = z+1 \end{array} \Rightarrow \frac{x-2}{1} = \frac{y+3}{-1} = \frac{z+1}{\gamma}$$
$$\vec{p}_a = (1, -1, \gamma)$$

$$\vec{p}_a \parallel \vec{p}_b \Rightarrow \exists s \in \mathbb{R}: \vec{p}_a = s \cdot \vec{p}_b$$
$$(1, -1, \gamma) = s \cdot k \cdot (-6+\beta, -2\alpha-1, \alpha\beta+3) \Rightarrow$$
$$\Rightarrow \frac{1}{-6+\beta} = \frac{-1}{-2\alpha-1} = \frac{\gamma}{\alpha\beta+3}$$

$6 - \beta = -2\alpha - 1$ $\quad$ $-6\gamma + \beta\gamma = \alpha\beta + 3$ $\quad$ $-2\alpha\gamma - \gamma = -\alpha\beta - 3$

$-\beta + 2\alpha = -7$ (a) $\quad$ $\alpha\beta - \beta\gamma + 6\gamma = -3$ (b) $\quad$ $\alpha\beta - 2\alpha\gamma - \gamma = -3$ (c)

(b)-(c): $-\beta\gamma + 2\alpha\gamma + 7\gamma = 0$
$(-\beta + 2\alpha)\gamma + 7\gamma = 0$

(b) i (c) su iste jednakosti

Uvrstimo (a) u (c). Imamo
$$\beta = 2\alpha + 7$$
$$2\alpha^2 + 7\alpha - 2\alpha\gamma - \gamma = -3$$
$$2\alpha^2 + (7-2\gamma)\alpha + 3 - \gamma = 0$$
$$D = (7-2\gamma)^2 - 8(3-\gamma) =$$
$$= 49 - 28\gamma + 4\gamma^2 - 24 + 8\gamma$$
$$= 4\gamma^2 - 20\gamma + 25 = (2\gamma - 5)^2$$

Kako je $\vec{p}_a \parallel \vec{p}_b \Rightarrow \vec{p}_a \times \vec{p}_b = 0$

$$\vec{p}_a \times \vec{p}_b = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & \gamma \\ -6+\beta & -2\alpha-1 & \alpha\beta+3 \end{vmatrix} = (0, 0, 0)$$

$$-2\alpha - 1 - 6 + \beta = 0$$
$$-2\alpha + \beta = 7$$
$$-\alpha\beta - 3 = -2\alpha\gamma - \gamma$$
$$\alpha\beta - 2\alpha\gamma - \gamma = -3$$
$$\alpha\beta + 3 + 6\gamma - \beta\gamma = 0$$
$$\alpha\beta - \beta\gamma + 6\gamma = -3$$

$$\alpha_{1,2} = \frac{2\gamma - 7 \pm (2\gamma - 5)}{4}$$

$$\alpha_1 = \frac{2\gamma - 7 - 2\gamma + 5}{4} = \frac{-2}{4} = -\frac{1}{2}$$

$$\alpha_2 = \frac{2\gamma - 7 + 2\gamma - 5}{4} = \frac{4\gamma - 12}{4} = \gamma - 3$$

$$2\left(\alpha + \frac{1}{2}\right)(\alpha - \gamma + 3) = 0$$

$$(2\alpha + 1)(\alpha - \gamma + 3) = 0$$

Ako bi α bilo $\alpha = -\frac{1}{2}$ tada bi imali da je $\beta = 6$ pa bi dobili da je $\vec{n}_2 = (0,0,0)$ što je nemoguće.

Pa je $\alpha - \gamma + 3 = 0$

$\alpha = \gamma - 3$ tj. $\gamma = \alpha + 3$

α ću odrediti na sljedeći način. Uzmimo tačku $A \in a$ i tačku $B \in b$. Tada $\overrightarrow{AB} \cdot \vec{n}_1 = 0$. ($a \parallel b$, $\vec{n}_1 \perp b$)

$A(2, -3, 1)$, $A \in a$

$B \in b$, ako uzmem $\gamma = 0$ imamo

$$\alpha x - z + 1 = 0 \quad (I)$$
$$x + 2z - 4 = 0 \quad (II)$$

(II) + 2(I): $x + 2\alpha x - 2 = 0$

$(1 + 2\alpha)x = 2$

$$x = \frac{2}{2\alpha + 1}$$

$z = \alpha x + 1$

$$z = \frac{2\alpha}{2\alpha + 1} + \frac{2\alpha + 1}{2\alpha + 1}$$

$$z = \frac{4\alpha + 1}{2\alpha + 1}$$

$$B\left(\frac{2}{2\alpha + 1}, 0, \frac{4\alpha + 1}{2\alpha + 1}\right)$$

$$\frac{2}{2\alpha+1} - \frac{4\alpha+2}{2\alpha+1}$$

$4\alpha + 1 - 2\alpha - 1$

$$\overrightarrow{AB} = \left(\frac{-4\alpha}{2\alpha + 1}, 3, \frac{2\alpha}{2\alpha + 1}\right)$$

$$\vec{n}_1 = (\alpha, -3, -1)$$

$\overrightarrow{AB} \cdot \vec{n}_1 = 0$ tj. $-\frac{4\alpha}{2\alpha + 1} \cdot \alpha + 3 \cdot (-3) + \frac{2\alpha}{2\alpha + 1} \cdot (-1) = 0 \quad / \cdot (2\alpha + 1)$

$-4\alpha^2 - 9(2\alpha + 1) - 2\alpha = 0$

$-4\alpha^2 - 20\alpha - 9 = 0$

$4\alpha^2 + 20\alpha + 9 = 0$

$D = 256$

$$\alpha_{1,2} = \frac{-20 \pm 16}{8} \Rightarrow \alpha_1 = -\frac{36}{8} \quad \alpha_2 = -\frac{4}{8}$$

$\alpha_1 = -\frac{9}{2}$ $\quad \alpha_2 = -\frac{1}{2}$

ovo rješenje otpada zbog $\alpha \neq -\frac{1}{2}$

Tražene konstante α, β i γ su

$$\alpha = -\frac{9}{2}, \quad \beta = -2 \quad i \quad \gamma = -\frac{3}{2}$$